

ON THE GEOMETRY OF BORDER RANK ALGORITHMS FOR $n \times 2$ BY 2×2 MATRIX MULTIPLICATION

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ABSTRACT. We make an in-depth study of the known border rank (i.e. approximate) algorithms for the matrix multiplication tensor $M_{(n,2,2)} \in \mathbb{C}^{2n} \otimes \mathbb{C}^4 \otimes \mathbb{C}^{2n}$ encoding the multiplication of an $n \times 2$ matrix by a 2×2 matrix.

1. INTRODUCTION

This is the first of a planned series of articles examining the geometry of algorithms for matrix multiplication tensors. Geometry has been used effectively in proving lower bounds for the complexity of matrix multiplication (see, e.g. [12, 8]), and one goal of this series is to initiate the use of geometry in proving upper bounds via practical algorithms for small matrix multiplication tensors.

A guiding principle is that if a tensor has symmetry, then there should be optimal expressions for it that reflect that symmetry. The matrix multiplication tensors have extraordinary symmetry. In this paper we examine algorithms, more precisely *border rank algorithms* (see below for the definition), that were originally found via numerical methods and computer searches.

Here is a picture illustrating the geometry of an algorithm due to Alekseev-Smirnov that we discuss in §7:

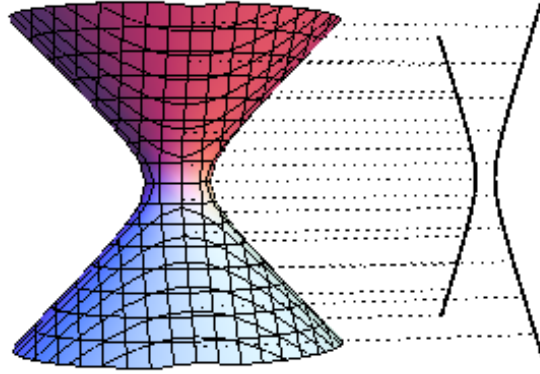


FIGURE 1. A quadric surface, a plane conic curve and a one parameter family of lines connecting them.

A tensor $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ has *rank one* if there exist $a \in \mathbb{C}^a$, $b \in \mathbb{C}^b$ and $c \in \mathbb{C}^c$ such that $T = a \otimes b \otimes c$. A *rank r expression* for a tensor $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ is a collection of rank one tensors

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T_1, \dots, T_r such that $T = T_1 + \dots + T_r$. A *border rank r algorithm* for T is an expression

$$T = \lim_{t \rightarrow 0} T_1(t) + \dots + T_r(t)$$

with each $T_j(t)$ of rank one, and for $t > 0$ the $T_j(t)$ are linearly independent. The first interesting border rank algorithm was found by Bini-Capovani-Lotti-Romani (BCLR) [2], essentially by accident: After Strassen's remarkable discovery [13] of a rank seven expression for the 2×2 matrix multiplication tensor, and Winograd's proof shortly afterward [15] that no rank six expression existed, BCLR attempted to determine if the rank of the 2×2 matrix multiplication tensor where an entry of one of the matrices is zero could have an expression of rank less than six. They used an alternating least squares iteration scheme on a computer. Instead of finding such an expression, they found the border rank expression (2) below. (That some tensors have border rank lower than rank was known to Terracini in 1911 [14], if not earlier, but not to the computer science community.) Later Smirnov [11, 10] and Alekseev-Smirnov [1], using similar, but more sophisticated methods, found further border rank algorithms for small matrix multiplication tensors.

In this paper we describe geometry in these algorithms, with very satisfactory answers in first few cases and successively weaker results as the tensors get larger. We begin, in §2 with a review of the matrix multiplication and BCLR-type tensors. We discuss the known upper and lower bounds on their border ranks in §3. In §4 and §5 we respectively discuss the geometry of border rank algorithms and the Segre variety. In sections §6–10 we analyze the various algorithms. We conclude with a brief discussion of the uniqueness of the BCLR algorithms in §11.

Notation. We let A, B, C, U, V, W denote complex vector spaces of dimensions $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{u}, \mathbf{v}, \mathbf{w}$. If W is a vector space then $\mathbb{P}W$ is the associated projective space of lines through the origin: $\mathbb{P}W = (W \setminus 0) / \sim$ where $w_1 \sim w_2$ if $w_1 = \lambda w_2$ for some nonzero complex number λ . Write $[w] \in \mathbb{P}W$ for the equivalence class of $w \in W \setminus 0$ and if $X \subset \mathbb{P}W$, let $\hat{X} \subset W$ denote the corresponding cone in W . The linear span of vectors w_1, \dots, w_s is denoted $\langle w_1, \dots, w_s \rangle$, and the span of $[w_j] \in \mathbb{P}W$ is similarly denoted $\langle [w_1], \dots, [w_s] \rangle \subset \mathbb{P}W$.

The set of rank one tensors in $\mathbb{P}(A \otimes B \otimes C)$ is isomorphic to $\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C$, and the inclusion into $\mathbb{P}(A \otimes B \otimes C)$ is denoted $\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A \otimes B \otimes C)$ and called the *Segre variety*.

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2. MATRIX MULTIPLICATION AND THE BCLRS TENSORS

The matrix multiplication tensor is

$$(1) \quad M_{\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle} = M_{\langle U, V, W \rangle} = \text{Id}_U \otimes \text{Id}_V \otimes \text{Id}_W \in (U^* \otimes V) \otimes (V^* \otimes W) \otimes (W^* \otimes U)$$

see [6, §2.5.2]. We write

$$M_{\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle} = \sum_{k=1}^{\mathbf{w}} \sum_{j=1}^{\mathbf{v}} \sum_{i=1}^{\mathbf{u}} x_j^i \otimes y_k^j \otimes z_i^k = \sum_{k=1}^{\mathbf{w}} \sum_{j=1}^{\mathbf{v}} \sum_{i=1}^{\mathbf{u}} (u^i \otimes v_j) \otimes (v^j \otimes w_k) \otimes (w^k \otimes u_i)$$

where $\{u_i\}$ is a basis of U with dual basis $\{u^i\}$ and similarly for V, W .

Define the generalized Bini-Capovani-Lotti-Romani-Smirnov tensor, corresponding to $m \times 2$ by 2×2 matrix multiplication with the x_1^1 entry set equal to zero:

$$T_{BCLRS,m} := M_{\langle m,2,2 \rangle} - x_1^1 \otimes (y_1^1 \otimes z_1^1 + y_2^1 \otimes z_1^2) \in \mathbb{C}^{2m-1} \otimes \mathbb{C}^4 \otimes \mathbb{C}^{2m}.$$

(In the original tensor BCLR set the x_1^2 entry to zero. We set x_1^1 equal to zero to facilitate comparisons between different values of m .)

3. BORDER RANK BOUNDS FOR THE BCLRS TENSORS

The following observation dates back to [2]:

Proposition 3.1. *If $\underline{\mathbf{R}}(T_{BCLRS,m}) = r$ and $\underline{\mathbf{R}}(T_{BCLRS,m'}) = r'$, then setting $n = m + m' - 1$, $\underline{\mathbf{R}}(M_{\langle n,2,2 \rangle}) \leq r + r'$.*

Upper bounds on the border ranks of these tensors are: $\underline{\mathbf{R}}(M_{\langle n,2,2 \rangle}) \leq 3n + \lceil \frac{n}{7} \rceil$ for all n , [2, 1, 11]. (Equality holds for $n = 1$ (classical), and $n = 2$ [5], see [4] for a better proof.)

It would be reasonable to expect that the BCLR, Alekseev-Smirnov, and Smirnov algorithms generalize to all m , so that $\underline{\mathbf{R}}(T_{BCLRS,m}) \leq 3m - 1$. If that happens, Proposition 3.1 would imply that $\underline{\mathbf{R}}(M_{\langle n,2,2 \rangle}) \leq 3n + 1$ for all n .

Proposition 3.2. $\underline{\mathbf{R}}(T_{BCLR}) = 5$ and for $m > 2$, $\underline{\mathbf{R}}(T_{BCLRS,m}) \geq 3m - 2$

Proof. The upper bound for T_{BCLR} comes from [2].

For the lower bounds we use Strassen's equations [12]: Let $T \in A \otimes B \otimes C$ be such that there exists $\alpha \in A^*$ with $T(\alpha) \in B \otimes C$ of maximal rank. Assuming $\mathbf{b} = \dim B \leq \dim C$, take $C' \subset C^*$ with $\dim C' = \mathbf{b}$, and such that $\text{rank} T(\alpha)|_{B^* \times C'} = \mathbf{b}$. Then use $T(\alpha)|_{B^* \otimes C'}$ to identify $B^* \otimes C' \simeq \text{End}(B)$, Strassen's equations state that for all $X_1, X_2 \in T(A^*)|_{\text{End}(B)}$, letting $[X_1, X_2]$ denote their commutator,

$$\underline{\mathbf{R}}(T) \geq \frac{1}{2} \text{rank}[X_1, X_2] + \mathbf{b}.$$

Consider

$$T_{BCLRS,m}(B^*) = \begin{pmatrix} y_1^2 & y_2^2 & & & & \\ & y_1^1 & y_2^1 & & & \\ & y_1^2 & y_2^2 & & & \\ & & & \ddots & & \\ & & & & y_1^1 & y_2^1 \\ & & & & y_1^2 & y_2^2 \end{pmatrix}$$

This is a $2m \times (2m - 1)$ matrix of linear forms. Take the submatrix setting the first column to zero to have a square matrix. Making generic choices, the first 1×1 block will not contribute to the commutator but all other blocks contribute a rank two matrix. We conclude $\underline{\mathbf{R}}(T_{BCLRS,m}) \geq \frac{1}{2}(2m - 2) + 2m - 1 = 3m - 2$. When $m = 2$ we can do a little better by considering $T_{BCLR}(A^*) \subset \mathbb{C}^4 \otimes \mathbb{C}^4$. Under generic choices the commutator of each 2×2 block contributes a rank one matrix and we conclude $\underline{\mathbf{R}}(T_{BCLR}) \geq \frac{1}{2}(2) + 4 = 5$. \square

Similarly, Strassen's equations imply $\underline{\mathbf{R}}(M_{\langle n,2,2 \rangle}) \geq 3n$. In summary:

$$3n \leq \underline{\mathbf{R}}(M_{\langle n,2,2 \rangle}) \leq 3n + \lceil \frac{n}{7} \rceil$$

and there is evidence for an upper bound of $3n + 1$.

4. WHAT IS A BORDER RANK ALGORITHM?

Usually a border rank algorithm is presented as

$$T = \lim_{t \rightarrow 0} T_1(t) + \cdots + T_r(t)$$

with each $T_j(t)$ of rank one and the $T_j(t)$ linearly independent when $t \neq 0$. To work geometrically we focus on the curve of r -planes $\langle T_1(t), \dots, T_r(t) \rangle \subset G(r, A \otimes B \otimes C)$ that the border rank algorithm defines. Here, for a vector space V , $G(r, V)$ denotes the Grassmannian of r -planes through the origin in V .

For the purposes of this paper, a border rank algorithm is a point $E \in G(r, A \otimes B \otimes C)$ such that $T \in E$ and there exists a curve E_t limiting to E with E_t spanned by r rank one elements for all $t > 0$.

Remark 4.1. More precisely a border rank algorithm should be thought of as an h -jet of a curve in the Grassmannian that is the h -jet of some curve spanned by rank one elements.

Remark 4.2. This discussion generalizes to arbitrary secant varieties, see [7].

A border rank algorithm will not be a rank algorithm when $T_1(0), \dots, T_r(0)$ fail to be linearly independent. Say this is the case and no subset of the points fails to be linearly independent. Then T can be any point in $\langle \hat{T}_{T_1(0)} \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), \dots, \hat{T}_{T_r(0)} \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \rangle$, where $\hat{T}_x \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset A \otimes B \otimes C$ denotes the affine tangent space to the Segre at x , see [6, §10.8]. In this case we call the algorithm *first order*.

A *second order algorithm* occurs when the sum of the tangent vectors fails to be linearly independent from the original r vectors (which themselves fail to be linearly independent). In this case, for each of the tangent vectors appearing, there is its image under the second fundamental form as described in Equation (3) below, and T is the sum of these vectors plus any point in the sum of the tangent spaces.

Higher order algorithms exist, but we do not discuss their geometry in this paper.

5. ON THE GEOMETRY OF THE SEGREG VARIETY

In order to have a border rank algorithm one must have r points on the Segre that fail to be linearly independent. The most naïve way to attain this is to have a point appearing at least twice. For example, the most classical tensor with border rank lower than rank is

$$a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 = \lim_{t \rightarrow 0} \frac{1}{t} [(a_1 + ta_2) \otimes (b_1 + tb_2) \otimes (c_1 + tc_2) - a_1 \otimes b_1 \otimes c_1]$$

where both points limit to $a_1 \otimes b_1 \otimes c_1$.

The next most naïve limits are when r points all lie on an $r - 1$ -plane. This is the case for Schönhage's algorithm for the sum of two disjoint tensors [9].

The configurations that arise in border rank algorithms for $T_{BCLRS, m}$ are more interesting. What follows are geometric preliminaries needed to describe them.

We first describe lines on Segre varieties. There are three types: α -lines, which are of the form $\mathbb{P}(\langle a_1, a_2 \rangle \otimes b \otimes c)$ for some $a_j \in A$, $b \in B$, $c \in C$, and the other two types are defined similarly and called β and γ lines.

Given two lines $L_\beta, L_\gamma \subset \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ respectively of type β, γ , if they do not intersect, then $\langle L_\beta, L_\gamma \rangle = \mathbb{P}^3$ and if the lines are general, furthermore $\langle L_\beta, L_\gamma \rangle \cap \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) = L_\beta \sqcup L_\gamma$.

However if $L_\beta = \mathbb{P}(a \otimes \langle b_1, b_2 \rangle \otimes c)$ and $L_\gamma = \mathbb{P}(a' \otimes b \otimes \langle c_1, c_2 \rangle)$ with $b \in \langle b_1, b_2 \rangle$ and $c \in \langle c_1, c_2 \rangle$, then they still span a \mathbb{P}^3 but $\langle L_\beta, L_\gamma \rangle \cap \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) = L_\beta \sqcup L_\gamma \sqcup L_\alpha$, where $L_\alpha = \mathbb{P}(\langle a, a' \rangle \otimes b \otimes c)$, and L_α intersects both L_β and L_γ .

Let $x, y, z \in \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ be distinct points that all lie on a line $L \subset \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. Then

$$\hat{T}_x \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \langle \hat{T}_y \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), \hat{T}_z \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \rangle.$$

In fact, the analogous statement is true for lines on any cominuscule variety, see [3, Lemma 3.3]. Because of this, it will be more geometrical to refer to $\hat{T}_L \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) := \langle \hat{T}_y \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), \hat{T}_z \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \rangle$, as the choice of $y, z \in L$ is irrelevant, at least for first order algorithms.

The matrix multiplication tensor $M_{(U,V,W)}$ (1) endows A, B, C with additional structure, e.g., $B = V^* \otimes W$, so there are two types of distinguished β -lines (corresponding to lines of rank one matrices), call them (β, ν^*) -lines and (β, ω) -lines, where, e.g., a ν^* -line is of the form $\mathbb{P}(a \otimes (\langle v^1, v^2 \rangle \otimes w) \otimes c)$, and among such lines there are further distinguished ones where moreover both a and c also have rank one. Call such further distinguished lines *special* (β, ν^*) -lines.

6. T_{BCLR}

Here $A \subset U^* \otimes V$ has dimension three, so we don't have the full space of 2×2 matrices.

What follows is a slight modification of the BCLR algorithm. We label the points such that x_1^1 is set equal to zero. The main difference is that in the original all five points moved, but here one is stationary.

$$\begin{aligned} p_1(t) &= x_2^1 \otimes (y_2^2 + y_1^2) \otimes (z_2^2 + tz_1^1) \\ p_2(t) &= -(x_2^1 - tx_2^2) \otimes y_2^2 \otimes (z_2^2 + t(z_1^1 + z_2^2)) \\ p_3(t) &= x_1^2 \otimes (y_1^2 + ty_2^1) \otimes (z_2^2 + z_2^1) \\ p_4(t) &= (x_1^2 - tx_2^2) \otimes (-y_1^2 + ty_1^1 - ty_2^1) \otimes z_2^1 \\ p_5(t) &= -(x_1^2 + x_2^1) \otimes y_1^2 \otimes z_2^2 \end{aligned}$$

and

$$(2) \quad T_{BCLR} = \frac{1}{t} [p_1(t) + \dots + p_5(t)].$$

Let $E^{BCLR} = \lim_{t \rightarrow 0} \langle p_1(t), \dots, p_5(t) \rangle \in G(5, A \otimes B \otimes C)$.

Theorem 6.1. *Notations as above. In the BCLR algorithm $E^{BCLR} \cap \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is the union of three lines: $L_{12,(\beta,\omega)}$, which is a special (β, ω) -line, $L_{21,(\gamma,\omega^*)}$, which is a special (γ, ω^*) -line, and L_α , which is an α -line with rank one b and c points. Moreover, the C -point of $L_{12,(\beta,\omega)}$ lies in the ω^* -line of $L_{21,(\gamma,\omega^*)}$, the B -point of $L_{21,(\gamma,\omega^*)}$ lies in the ω -line of $L_{12,(\beta,\omega)}$ and L_α is the unique line on the Segre intersecting $L_{12,(\beta,\omega)}$ and $L_{21,(\gamma,\omega^*)}$ (and thus it is contained in their span).*

Explicitly:

$$\begin{aligned} L_{12,(\beta,\omega)} &= x_2^1 \otimes (v^2 \otimes W) \otimes z_2^1 \\ L_{21,(\gamma,\omega^*)} &= x_1^2 \otimes y_2^2 \otimes (W^* \otimes u_2) \\ L_\alpha &= \langle x_1^2, x_2^1 \rangle \otimes y_2^2 \otimes z_2^1. \end{aligned}$$

Furthermore, $E^{BCLR} = \langle T_{BCLR}, L_{12,(\beta,\omega)}, L_{21,(\gamma,\omega^*)} \rangle$ and

$$T_{BCLR} \in \langle \hat{T}_{L_{12,(\beta,\omega)}} \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), \hat{T}_{L_{21,(\gamma,\omega^*)}} \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \rangle.$$

Proof. Write $p_j = p_j(0)$. Then (up to sign, which is irrelevant for geometric considerations)

$$\begin{aligned} p_1 &= x_2^1 \otimes (y_2^2 + y_1^2) \otimes z_2^2 \\ p_2 &= x_2^1 \otimes y_2^2 \otimes z_2^2 \\ p_3 &= x_1^2 \otimes y_1^2 \otimes (z_2^2 + z_2^1) \\ p_4 &= x_1^2 \otimes y_1^2 \otimes z_2^1 \\ p_5 &= (x_1^2 + x_2^1) \otimes y_1^2 \otimes z_2^2 \end{aligned}$$

The configuration of lines is as follows:

$$\begin{aligned} p_1, p_2 &\in x_2^1 \otimes (v^2 \otimes W) \otimes z_2^2 \\ p_3, p_4 &\in x_1^2 \otimes y_1^2 \otimes (W^* \otimes u_2) \\ p_5 &\in (x_2^1, x_1^2) \otimes y_1^2 \otimes z_2^2. \end{aligned}$$

To see there are no other points in $E^{BCLR} \cap \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, first note that any such point would have to lie on $\text{Seg}(\mathbb{P}\langle x_2^1, x_1^2 \rangle \times \mathbb{P}\langle y_1^2, y_2^2 \rangle \times \mathbb{P}\langle z_2^1, z_2^2 \rangle)$ because there is no way to eliminate the rank two $x_2^2 \otimes (y_1^2 \otimes z_2^1 + y_2^2 \otimes z_2^2)$ term in T_{BCLR} with a linear combination of p_1, \dots, p_4 . Let $[(sx_2^1 + tx_1^2) \otimes (uy_2^2 + vy_1^2) \otimes (pz_2^2 + qz_2^1)]$ be an arbitrary point on this variety. To have it be in the span of p_1, \dots, p_4 it must satisfy the equations $suq = 0$, $svq = 0$, $tuq = 0$, $tup = 0$. Keeping in mind that one cannot have $(s, t) = (0, 0)$, $(u, v) = (0, 0)$, or $(p, q) = (0, 0)$, we conclude the only solutions are the three lines already exhibited.

We have

$$\begin{aligned} p_1(0)' &= x_2^1 \otimes (y_2^2 + y_1^2) \otimes z_1^1 \\ p_2(0)' &= x_2^2 \otimes y_2^2 \otimes z_2^2 - x_2^1 \otimes y_2^2 \otimes (-z_1^2 + z_1^1) \\ p_3(0)' &= x_1^2 \otimes y_2^1 \otimes (z_2^2 + z_2^1) \\ p_4(0)' &= x_2^2 \otimes y_1^2 \otimes z_2^1 + x_1^2 \otimes (y_1^1 - y_2^1) \otimes z_2^1 \\ p_5(0)' &= 0 \end{aligned}$$

Then $T_{BCLR} = (p'_1 + p'_2) + (p'_3 + p'_4)$ where $p'_1 + p'_2 \in T_{L_{12}, (\beta, \omega)} \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ and $p'_3 + p'_4 \in T_{L_{21}, (\gamma, \omega^*)} \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. \square

Remark 6.2. When we allow $GL_4^{\times 3} \rtimes \mathfrak{S}_3$ to act on $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, the subgroup preserving the tensor $M_{(2,2,2)}$ is $SL_2 \times SL_2 \times SL_2 \rtimes \mathbb{Z}_3$, where the SL_2 's are respectively $SL(U), SL(V), SL(W)$ and the \mathbb{Z}_3 is most easily seen by viewing matrix multiplication as a trilinear map which sends three matrices to the trace of their product: $M, N, L \mapsto \text{Tr}(MNL)$. The \mathbb{Z}_3 -symmetry follows as $\text{Tr}(MNL) = \text{Tr}(NLM)$. The full symmetry group includes a $\mathbb{Z}_2 \rtimes \mathbb{Z}_3$ action as $\text{Tr}(X) = \text{Tr}(X^T)$, where X^T denotes the transpose of X . By removing x_1^1 from our tensor, we lose the \mathbb{Z}_3 , but retain a \mathbb{Z}_2 action which corresponds to $\text{Tr}(MNL) = \text{Tr}(M^T L^T N^T)$. Similarly we lose our $GL(U) \times GL(V)$ symmetry but retain our $GL(W)$ action. By composing our discrete \mathbb{Z}_2 symmetry with another \mathbb{Z}_2 action which switches the basis vectors of W , the action swaps $p_1(t) + p_2(t)$ with $p_3(t) + p_4(t)$ and $L_{12}, (\beta, \omega)$ with $L_{21}, (\gamma, \omega^*)$. This \mathbb{Z}_2 action fixes $p_5(t)$.

Remark 6.3. Note that it is important that p_5 lies neither on $L_{12}, (\beta, \omega)$ nor on $L_{21}, (\gamma, \omega^*)$, so that no subset of the five points lies in a linearly degenerate position to enable us to have tangent vectors coming from all five points, but we emphasize that any such point (i.e., any point on

the line L_α not on the original lines) would have worked equally well, so the geometric object is this configuration of lines.

7. $T_{BCLRS,3}$

Here is the algorithm in [1, Thm. 2] only changing the element set to zero to x_1^1 (it is $x_2^3 = a_2^3$ in [1]).

$$\begin{aligned}
p_1(t) &= \left(\frac{-1}{2}t^2x_2^3 - \frac{1}{2}tx_1^2 + x_1^2\right) \otimes (-y_1^2 + y_2^2 + ty_1^1) \otimes (z_3^1 + tz_2^1) \\
p_2(t) &= (x_1^2 + \frac{1}{2}x_2^1) \otimes (y_1^2 - y_2^2) \otimes (z_3^1 + z_3^2 + tz_2^1 + tz_2^2) \\
p_3(t) &= (t^2x_2^3 + tx_1^3 - \frac{1}{2}tx_2^2 - x_1^2) \otimes (y_1^2 + y_2^2 + ty_2^1) \otimes z_3^2 \\
p_4(t) &= \left(\frac{1}{2}t^2x_2^3 - tx_1^3 - \frac{1}{2}tx_2^2 + x_1^2\right) \otimes (y_1^2 + y_2^2 - ty_1^1) \otimes z_3^1 \\
p_5(t) &= (-t^2x_2^3 + tx_2^2 - x_1^2) \otimes y_1^2 \otimes (z_3^2 + \frac{1}{2}tz_2^1 + \frac{1}{2}tz_2^2 - t^2z_1^1) \\
p_6(t) &= \left(\frac{1}{2}tx_2^2 + x_1^2\right) \otimes (-y_1^2 + y_2^2 + ty_2^1) \otimes (z_3^2 + tz_2^2) \\
p_7(t) &= (-tx_1^3 + x_1^2 + \frac{1}{2}x_2^1) \otimes (y_1^2 + y_2^2) \otimes (-z_3^1 + z_3^2) \\
p_8(t) &= (tx_2^2 + x_1^2) \otimes y_2^2 \otimes (z_3^1 + \frac{1}{2}tz_2^1 + \frac{1}{2}tz_2^2 + t^2z_1^1)
\end{aligned}$$

Then

$$T_{BCLRS,3} = \frac{1}{t^2}[p_1(t) + \cdots + p_8(t)].$$

Let $E^{AS,3} = \lim_{t \rightarrow 0} \langle p_1(t), \dots, p_8(t) \rangle \in G(8, A \otimes B \otimes C)$.

Theorem 7.1. *Notations as above. In the Alekseev-Smirnov algorithm for $T_{BCLRS,3}$, $E^{AS,3} \cap \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is the union of two irreducible algebraic surfaces, both abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$: The first is a sub-Segre variety:*

$$\text{Seg}_{21,(\beta,\omega),(\gamma,\omega^*)} := [x_1^2] \times \mathbb{P}(v^2 \otimes W) \times \mathbb{P}(W^* \otimes u_3),$$

The second, \mathbb{L}_α is a union of lines passing through $\text{Seg}_{21,(\beta,\omega),(\gamma,\omega^)}$ and the plane conic curve:*

$$C_{12,(\beta,\omega),(\gamma,\omega^*)} := \mathbb{P}(\cup_{[s,t] \in \mathbb{P}^1} x_2^1 \otimes (sy_1^2 - ty_2^2) \otimes (sz_3^2 + tz_3^1)).$$

The three varieties $C_{12,(\beta,\omega),(\gamma,\omega^)}$, $\text{Seg}_{21,(\beta,\omega),(\gamma,\omega^*)}$, and \mathbb{L}_α respectively play roles analogous to the lines $L_{12,(\beta,\omega)}$, $L_{21,(\gamma,\omega^*)}$, and L_α , as described below.*

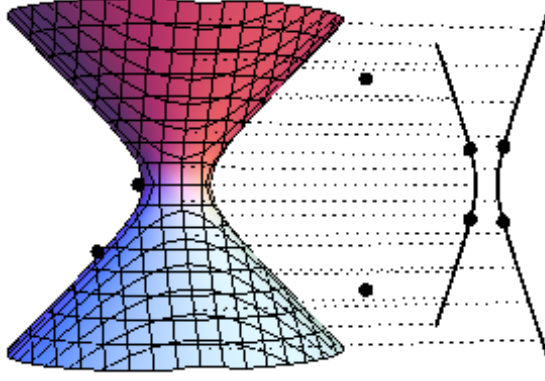


FIGURE 2. The curve $C_{12,(\beta,\omega),(\gamma,\omega^*)}$ with its four points, the surface $Seg_{21,(\beta,\omega),(\gamma,\omega^*)}$, with its four points (only two of which are visible), and the surface \mathbb{L}_α with its two points which don't lie on either the curve or surface $Seg_{21,(\beta,\omega),(\gamma,\omega^*)}$.

Proof. The limit points are (up to sign):

$$p_1 = x_1^2 \otimes (y_1^2 - y_2^2) \otimes z_3^1$$

$$p_3 = x_1^2 \otimes (y_1^2 + y_2^2) \otimes z_3^2$$

$$p_4 = x_1^2 \otimes (y_1^2 + y_2^2) \otimes z_3^1$$

$$p_6 = x_1^2 \otimes (y_1^2 - y_2^2) \otimes z_3^2$$

$$p_5 = x_2^1 \otimes y_1^2 \otimes z_3^2$$

$$p_8 = x_2^1 \otimes y_2^2 \otimes z_3^1$$

$$p_2 = (x_1^2 + \frac{1}{2}x_2^1) \otimes (y_1^2 - y_2^2) \otimes (z_3^1 + z_3^2)$$

$$p_7 = (x_1^2 + \frac{1}{2}x_2^1) \otimes (y_1^2 + y_2^2) \otimes (z_3^1 - z_3^2)$$

Just as with T_{BCLR} , the limit points all lie on a $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, in fact the “same” $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. Pictorially the Segres are:

$$\begin{pmatrix} 0 & * \\ * & \end{pmatrix} \times \begin{pmatrix} & \\ * & * \end{pmatrix} \times \begin{pmatrix} * \\ * \end{pmatrix}$$

for $T_{BCLRS,2}$ and

$$\begin{pmatrix} 0 & * \\ * & \end{pmatrix} \times \begin{pmatrix} & \\ * & * \end{pmatrix} \times \begin{pmatrix} * \\ * \end{pmatrix}$$

for $T_{BCLRS,3}$. Here $E^{AS,3} \cap Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is the union of a one-parameter family of lines \mathbb{L}_α passing through a plane conic and a special $\mathbb{P}^1 \times \mathbb{P}^1$: $Seg_{21,(\beta,\omega),(\gamma,\omega^*)} := [x_1^2] \times \mathbb{P}(v^2 \otimes W) \times \mathbb{P}(W^* \otimes u_3)$ (which contains p_1, p_3, p_4, p_6). To define the family and make the similarity with the

BCLR case clearer, first define the plane conic curve

$$C_{12,(\beta,\omega),(\gamma,\omega^*)} := \mathbb{P}(\cup_{[s,t] \in \mathbb{P}^1} x_2^1 \otimes (sy_1^2 - ty_2^2) \otimes (sz_3^2 + tz_3^1)).$$

The points p_5, p_8 lie on this conic (respectively the values $(s, t) = (1, 0)$ and $(s, t) = (0, 1)$). Then define the variety

$$\mathbb{L}_\alpha := \mathbb{P}(\cup_{[\sigma,\tau] \in \mathbb{P}^1} \cup_{[s,t] \in \mathbb{P}^1} (\sigma x_2^1 + \tau x_1^2) \otimes (sy_1^2 - ty_2^2) \otimes (sz_3^2 + tz_3^1)),$$

which is a one-parameter family of lines intersecting the conic and the special $\mathbb{P}^1 \times \mathbb{P}^1$. The points p_2, p_7 lie on \mathbb{L}_α but not on the conic. Explicitly p_2 (resp. p_7) is the point corresponding to the values $(\sigma, \tau) = (1, \frac{1}{2})$ and $(s, t) = (1, 1)$ (resp. $(s, t) = (1, -1)$).

The analog of L_α in the T_{BCLR} algorithm is \mathbb{L}_α , and $C_{12,(\beta,\omega),(\gamma,\omega^*)}$ and $Seg_{21,(\beta,\omega),(\gamma,\omega^*)}$ are the analogs of the lines $L_{12,(\beta,\omega)}, L_{21,(\gamma,\omega^*)}$. (A difference here is that $C_{12,(\beta,\omega),(\gamma,\omega^*)} \subset \mathbb{L}_\alpha$.)

The span of the configuration is the span of a \mathbb{P}^2 (the span of the conic) and a \mathbb{P}^3 (the span of the $\mathbb{P}^1 \times \mathbb{P}^1$), i.e., a \mathbb{P}^6 .

The proof that these are the only points in the intersection is similar to the BCLR case. \square

Remark 7.2. We expect that just as with T_{BCLR} , the particular 8 points in this configuration one uses in the limit are irrelevant as long as they are sufficiently general that no seven of them fail to be linearly independent.

The tangent vectors to a point $[a \otimes b \otimes c] \in Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ are of the form $a' \otimes b \otimes c + a \otimes b' \otimes c + a \otimes b \otimes c'$. The following chart gives the vectors (a', b', c') for the tangent vectors that appear in the algorithm. Blank spaces correspond to a zero vector:

$$\begin{array}{llll} p_1 : & -\frac{1}{2}x_2^2 & y_1^1 & z_2^1 \\ p_3 : & x_1^3 - \frac{1}{2}x_2^2 & y_2^1 & \\ p_4 : & -x_1^3 - \frac{1}{2}x_2^2 & -y_1^1 & \\ p_6 : & \frac{1}{2}x_2^2 & y_2^1 & z_2^2 \\ \\ p_5 : & x_2^2 & & \frac{1}{2}z_2^1 + \frac{1}{2}z_2^2 \\ p_8 : & x_2^2 & & \frac{1}{2}z_2^1 + \frac{1}{2}z_2^2 \\ \\ p_2 : & & & z_2^1 + z_2^2 \\ p_7 : & -x_1^3 & & . \end{array}$$

There are two types of points that can appear at second order: ordinary tangent vectors, and vectors arising from the *second fundamental form*. The latter must appear: if a tangent vector $a' \otimes b \otimes c + a \otimes b' \otimes c + a \otimes b \otimes c'$ appears at first order, then the vector

$$(3) \quad a' \otimes b' \otimes c + a \otimes b' \otimes c' + a' \otimes b \otimes c'$$

must appear at second order, see [3]. The following chart gives the new ordinary tangent vectors appearing at second order in the same format as the tangent vectors above:

$$\begin{array}{ll}
p_1 : & \frac{-1}{2}x_2^3 \\
p_3 : & x_2^3 \\
p_4 : & \frac{1}{2}x_2^3 \\
p_6 : & \\
\\
p_5 : & -x_2^3 \quad -z_1^1 \\
p_8 : & z_1^2 \\
\\
p_2 : & \\
p_7 : & .
\end{array}$$

Pictorially, the order entries are reached at (which coincides with the expression for T_{BCLR} when one truncates) is

$$\begin{pmatrix} X & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}.$$

Explicitly:

$$\begin{aligned}
p_1(0)' &= \frac{-1}{2}x_2^2 \otimes (-y_1^2 + y_2^2) \otimes z_3^1 + x_1^2 \otimes y_1^1 \otimes z_3^1 + x_1^2 \otimes (-y_1^2 + y_2^2) \otimes z_2^1 \\
p_3(0)' &= (x_1^3 - \frac{1}{2}x_2^2) \otimes (y_1^2 + y_2^2) \otimes z_3^2 + x_1^2 \otimes y_2^1 \otimes z_3^2 \\
p_4(0)' &= (-x_1^3 - \frac{1}{2}x_2^2) \otimes (y_1^2 + y_2^2) \otimes z_3^1 + x_1^2 \otimes -y_1^1 \otimes z_3^1 \\
p_6(0)' &= \frac{1}{2}x_2^2 \otimes (-y_1^2 + y_2^2) \otimes z_3^2 + x_1^2 \otimes y_2^1 \otimes z_3^2 + x_1^2 \otimes (-y_1^2 + y_2^2) \otimes z_2^2 \\
\\
p_5(0)' &= x_2^2 \otimes y_1^2 \otimes z_3^2 - x_2^1 \otimes y_1^2 \otimes (\frac{1}{2}z_2^1 + \frac{1}{2}z_2^2) \\
p_7(0)' &= (-x_1^3 + \frac{1}{2}x_2^1) \otimes (y_1^2 + y_2^2) \otimes (-z_3^1 + z_3^2) \\
\\
p_2(0)' &= (x_1^2 + \frac{1}{2}x_2^1) \otimes (y_1^2 - y_2^2) \otimes (z_2^1 + z_2^2) \\
p_8(0)' &= x_2^2 \otimes y_2^2 \otimes z_3^1 + x_2^1 \otimes y_2^2 \otimes (\frac{1}{2}z_2^1 + \frac{1}{2}z_2^2)
\end{aligned}$$

We split the t^2 coefficients into the two types discussed above: the second fundamental form terms, in the following table, and the tangent vectors appearing at second order, which are in the table below.

$$\begin{array}{ll}
p_1 & \frac{-1}{2}x_2^2 \otimes y_1^1 \otimes z_3^1 + x_1^2 \otimes y_1^1 \otimes z_2^1 + \frac{-1}{2}x_2^2 \otimes (-y_1^2 + y_2^2) \otimes z_2^1 \\
p_3 & (x_2^3 - \frac{1}{2}x_2^2) \otimes y_2^1 \otimes z_3^2 \\
p_4 & (-x_1^3 - \frac{1}{2}x_2^2) \otimes -y_1^1 \otimes z_3^1 \\
p_6 & \frac{1}{2}x_2^2 \otimes y_2^1 \otimes z_3^2 + x_1^2 \otimes y_2^1 \otimes z_2^2 + \frac{1}{2}x_2^2 \otimes (-y_1^2 + y_2^2) \otimes z_2^2 \\
\\
p_5 & x_2^2 \otimes y_1^2 \otimes (\frac{1}{2}z_2^1 + \frac{1}{2}z_2^2) \\
p_7 & \\
\\
p_2 & \\
p_8 & x_2^2 \otimes y_2^2 \otimes (\frac{1}{2}z_2^1 + \frac{1}{2}z_2^2)
\end{array}$$

$$\begin{array}{ll}
p_1 & \frac{-1}{2}x_2^3 \otimes (-y_1^2 + y_2^2) \otimes z_3^1 \\
p_3 & x_2^3 \otimes (y_1^2 + y_2^2) \otimes z_3^2 \\
p_4 & \frac{1}{2}x_2^3 \otimes (y_1^2 + y_2^2) \otimes z_3^1 \\
p_6 & \\
p_5 & -x_2^3 \otimes y_1^2 \otimes z_3^2 + -x_2^1 \otimes y_1^2 \otimes -z_1^1 \\
p_7 & \\
p_2 & \\
p_8 & x_2^1 \otimes y_2^2 \otimes z_1^2.
\end{array}$$

Then $T_{BCLRS,3}$ is the sum of the terms in the two tables above.

8. $T_{BCLRS,4}$

This algorithm is more complicated and qualitatively different than the others, so we only discuss it briefly. Note that here the order of the sizes of the matrices are changed: 2×4 , 4×2 and 2×2 .

$$\begin{aligned}
p_1(t) &= (x_2^1 - \frac{7}{25}t^3x_3^1 + x_2^2 - \frac{1}{50}t^3x_3^2 + t^2x_4^2) \otimes (\frac{-47}{112}t^4y_2^2 + \frac{25}{7}ty_2^3 + \frac{8}{21}y_1^4 + t^2y_2^4) \otimes (-z_2^1 + t^2\frac{1}{2}z_1^2 + z_2^2) \\
p_2(t) &= (x_2^1 + \frac{1}{8}x_2^2 - \frac{1}{50}t^3x_3^2 + \frac{1}{8}t^2x_4^2) \otimes (\frac{8}{7}t^4y_2^2 + \frac{3200}{63}ty_2^3 + \frac{128}{189}y_1^4) \otimes (t^2\frac{21}{16}z_1^1 + \frac{1}{8}z_2^1 + \frac{13}{16}t^2z_1^2 - z_2^2) \\
p_3(t) &= (x_2^1 - \frac{103}{300}x_3^1x_3^1 + x_2^2) \otimes (\frac{1}{8}t^5y_2^1 + \frac{1}{3}t^3b_1^2 + \frac{25}{16}t^5y_2^2 + \frac{5}{4}y_1^3 - \frac{1}{3}ty_1^3 - t^3y_2^4) \otimes (\frac{1}{2}t^2z_1^2 + z_2^2) \\
p_4(t) &= (-x_2^1 + x_2^2 + \frac{1}{50}tx_3^2 + x_4^2) \otimes (-\frac{25}{9}ty_2^3 + \frac{8}{27}y_1^4) \otimes (3t^2z_1^1 - z_2^1 + \frac{1}{2}t^2z_1^2 - z_2^2) \\
p_5(t) &= (x_2^1 + \frac{1}{50}t^3x_3^1) \otimes (t^3y_1^2 + \frac{75}{32}y_1^3 - 75t^2y_2^3 - ty_1^4) \otimes (t^2z_1^1 + \frac{2}{3}t^2z_1^2 - \frac{2}{3}z_2^2) \\
p_6(t) &= (-\frac{1}{8}x_2^1 + \frac{61}{800}t^3x_3^1 + x_1^2 - \frac{1}{8}x_2^2) \otimes (5y_1^3 + t^5y_2^1) \otimes (\frac{5}{2}t^2z_1^1 + t^2z_1^2 + z_2^2) \\
p_7(t) &= (x_2^1 + \frac{3}{100}t^3x_3^1 + t^2x_4^1 - x_2^2 - t^2x_4^2) \otimes (-\frac{7}{16}t^4y_2^2 + \frac{1}{3}y_1^4) \otimes (3t^2z_1^1 - z_2^1) \\
p_8(t) &= (x_2^1 + \frac{19}{300}t^3x_3^1 - x_2^2) \otimes (\frac{1}{3}x^3y_1^2 + \frac{5}{4}y_1^3 - \frac{1}{3}ty_1^4) \otimes (-\frac{5}{2}tz_1^2 + z_2^2) \\
p_9(t) &= (x_2^1 - \frac{29}{100}t^3x_3^1 + t^2x_4^1 + x_2^2 + t^2x_4^2) \otimes (\frac{1}{3}y_1^4 + t^2y_2^4) \otimes (z_2^1) \\
p_{10}(t) &= (-\frac{5}{16}x_2^1 - 5x_1^2 + \frac{5}{8}x_2^2) \otimes y_1^3 \otimes (\frac{5}{2}t^2z_1^1 - 5t^2z_1^2 + z_2^2) \\
p_{11}(t) &= (-x_1^2 + \frac{1}{30}t^3x_3^2) \otimes (-t^3y_1^1 + 30y_1^3) \otimes z_1^2.
\end{aligned}$$

Then

$$T_{BCLRS,4} = \frac{1}{t^5}[p_3(t) + p_5(t) + p_6(t) + p_{10}(t) + t(p_1(t) + p_2(t) + p_4(t) + p_7(t) + p_9(t)) + t^2p_{11}(t)].$$

The limit points are (ignoring scales which are irrelevant for the geometry):

$$\begin{aligned}
p_1 &= (x_2^1 + x_2^2) \otimes y_1^4 \otimes (z_2^1 - z_2^2) \\
p_2 &= (x_2^1 + \frac{1}{8}x_2^2 + \frac{1}{8}x_4^2) \otimes y_1^4 \otimes (\frac{1}{8}z_2^1 - z_2^2) \\
p_3 &= x_2^1 + x_2^2 \otimes y_1^3 \otimes z_2^2 \\
p_4 &= -x_2^1 + x_2^2 \otimes y_1^4 \otimes (z_2^1 + z_2^2) \\
p_5 &= x_2^1 \otimes y_1^3 \otimes z_2^2 \\
p_6 &= (-x_2^1 + 8x_1^2 - x_2^2) \otimes y_1^3 \otimes z_2^2 \\
p_7 &= (x_2^1 - x_2^2) \otimes y_1^4 \otimes z_2^1 \\
p_8 &= (x_2^1 - x_2^2) \otimes y_1^3 \otimes z_2^2 \\
p_9 &= (x_2^1 + x_2^2) \otimes y_1^4 \otimes z_2^1 \\
p_{10} &= (-x_2^1 - 16x_1^2 + 2x_2^2) \otimes y_1^3 \otimes z_2^2 \\
p_{11} &= x_1^2 \otimes y_1^3 \otimes z_1^2.
\end{aligned}$$

Here p_3, p_5, p_6, p_{10} (the “honest” limit points) lie on a $\mathbb{P}^2 \times \mathbb{P}^0 \times \mathbb{P}^0$, namely $\mathbb{P}(\langle x_2^1, x_2^2, x_1^2 \rangle \otimes y_1^3 \otimes z_2^2)$, a much simpler limit configuration than previously. The point x_2^2 shows up at zero-th order, whereas in the previous algorithms only vectors tangent to x_1^1 in $\text{Seg}(\mathbb{P}U^* \otimes \mathbb{P}V)$ showed up at zero-th order. The high order of the algorithm makes its geometry difficult to analyze.

9. THE ALEKSEEV-SMIRNOV BORDER RANK ALGORITHM FOR $M_{(4,2,2)}$

While this algorithm does not split the matrix multiplication tensor into the sum of two tensors and two algorithms, it still has features of the other algorithms.

We rearrange the points and flip the super/subscript of z (the ordering in [1] was 1,2,3,4,5,10,13,6,11,7,9,8,12). We also modified the derivatives of p_{12} and p_{13} , and the second derivative of p_4 .

$$\begin{aligned}
p_1(t) &= (-tx_2^1 + x_2^2 + x_2^4) \otimes (-y_1^1 + y_2^2) \otimes (-z_1^1 + z_3^1 - tz_4^1 + t^2 z_2^2) \\
p_2(t) &= (tx_2^1 - x_2^2) \otimes (-ty_1^2 + y_2^2) \otimes (-z_1^1 + z_3^1 - tz_4^1 + tz_1^2) \\
p_3(t) &= (t^2 x_1^1 + tx_2^1 - tx_1^2 - x_2^2 - x_2^4) \otimes y_1^1 \otimes (z_1^1 - t^2 z_2^2) \\
p_4(t) &= (t^2 x_1^3 - tx_1^4 - tx_2^1 + x_2^2 + x_2^4) \otimes (-y_1^1 + ty_1^2) \otimes (-z_3^1 + tz_4^1) \\
p_5(t) &= x_2^4 \otimes y_2^2 \otimes (z_1^1 - z_3^1 + tz_4^1 - t^2 z_2^2 - tz_3^2 + t^2 z_4^2) \\
\\
p_6(t) &= (t^2 x_1^1 + x_2^2) \otimes (y_2^1 - t^2 y_1^2 + ty_2^2) \otimes z_2^1 \\
p_7(t) &= x_2^2 \otimes y_2^1 \otimes (-z_1^1 - tz_1^2 - z_2^1) \\
\\
p_8(t) &= x_1^2 \otimes (-ty_1^1 + ty_1^2 + y_2^1) \otimes (-z_1^1 - tz_2^1 + t^2 z_2^2) \\
p_9(t) &= (x_1^2 + x_2^2) \otimes (y_2^1 + ty_1^2) \otimes (z_1^1 + tz_1^2) \\
\\
p_{10}(t) &= (t^2 x_1^3 + tx_2^3 + x_2^4) \otimes (y_2^1 - ty_1^2) \otimes (-z_3^1 + z_3^2) \\
p_{11}(t) &= (tx_2^3 + x_2^4) \otimes (-y_2^1 + ty_1^2 + ty_2^2) \otimes z_3^2 \\
\\
p_{12}(t) &= (t^2 x_1^3 + x_1^4) \otimes (ty_1^1 + y_2^1 - t^2 y_1^2) \otimes (z_3^1 + t^2 z_4^2) \\
p_{13}(t) &= (tx_2^3 + x_2^4 - x_1^4) \otimes y_2^1 \otimes z_3^1.
\end{aligned}$$

Then $M_{\langle 2,2,4 \rangle} = \lim_{t \rightarrow 0} \frac{1}{t^2} \sum p_i(t)$.

The limiting points are (ignoring signs irrelevant for geometry):

$$\begin{aligned}
p_1 &= (x_2^2 + x_2^4) \otimes (y_1^1 - y_2^2) \otimes (z_1^1 - z_3^1) \\
p_2 &= x_2^2 \otimes y_2^2 \otimes (z_1^1 - z_3^1) \\
p_3 &= (x_2^2 + x_2^4) \otimes y_1^1 \otimes z_1^1 \\
p_4 &= (x_2^2 + x_2^4) \otimes y_1^1 \otimes z_3^1 \\
p_5 &= x_2^4 \otimes y_2^2 \otimes (z_1^1 - z_3^1) \\
\\
p_6 &= x_2^2 \otimes y_2^1 \otimes z_1^2 \\
p_7 &= x_2^2 \otimes y_2^1 \otimes (z_1^1 + z_1^2) \\
p_8 &= x_1^2 \otimes y_2^1 \otimes z_1^1 \\
p_9 &= x_1^2 + x_2^2 \otimes y_2^1 \otimes z_1^1 \\
\\
p_{10} &= x_2^4 \otimes y_2^1 \otimes (z_3^1 - z_3^2) \\
p_{11} &= x_2^4 \otimes y_2^1 \otimes z_3^2 \\
p_{12} &= x_1^4 \otimes y_2^1 \otimes z_3^1 \\
p_{13} &= (x_2^4 - x_1^4) \otimes y_2^1 \otimes z_3^1.
\end{aligned}$$

The terms are grouped as above because there are three independent failures of linear independence: First $\langle p_1, \dots, p_5 \rangle \cap \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ form a BCLR-type configuration of special (γ, μ) and (α, μ^*) lines plus a β -line with rank one A, C elements that intersects the special lines, namely

$$\begin{aligned} L_{(\gamma, \mu)} &= (x_2^2 + x_2^4) \otimes y_1^1 \otimes (w^1 \otimes \langle u_1, u_3 \rangle) \\ L_{(\alpha, \mu^*)} &= (\langle u^2, u^4 \rangle \otimes v_2) \otimes y_2^2 \otimes (z_1^1 - z_3^1) \\ L_\beta &= (x_2^2 + x_2^4) \otimes \langle y_1^1, y_2^2 \rangle \otimes (z_1^1 - z_3^1). \end{aligned}$$

In this configuration, the space B plays the role of A in the earlier expressions. Then there are two pairs of lines that intersect in a point causing linear dependence (subscript indicates type). They are $\langle p_6, \dots, p_9 \rangle$ and $\langle p_{10}, \dots, p_{13} \rangle$, which are each contained in a \mathbb{P}^2 spanned by two intersecting lines on the Segre.

$$\begin{aligned} S_{(\gamma, \omega^*)} &= x_2^2 \otimes y_2^1 \otimes (W^* \otimes u_1) \\ S_{(\alpha, \nu)} &= (u^2 \otimes V) \otimes y_2^1 \otimes z_1^1 \end{aligned}$$

and

$$\begin{aligned} T_{(\gamma, \omega^*)} &= x_2^4 \otimes y_2^1 \otimes (W^* \otimes u_3) \\ T_{(\alpha, \nu)} &= (u^4 \otimes V) \otimes y_2^1 \otimes z_3^1. \end{aligned}$$

We use the same notation as above in describing the first and second derivatives. The first derivatives correspond to:

$$\begin{array}{cccc} p'_1 & -x_2^1 & & -z_4^1 \\ p'_2 & x_2^1 & -y_1^2 & -z_4^1 + z_1^2 \\ p'_3 & x_2^1 - x_1^2 & & \\ p'_4 & -x_1^4 - x_2^1 & y_1^2 & z_4^1 \\ p'_5 & & & z_4^1 - z_3^2 \\ p'_6 & & y_2^2 & \\ p'_7 & & & -z_2^1 \\ p'_8 & & -y_1^1 + y_1^2 & -z_2^1 \\ p'_9 & & y_1^2 & z_2^1 \\ p'_{10} & x_2^3 & -y_1^2 & \\ p'_{11} & x_2^3 & y_1^2 + y_2^2 & \\ p'_{12} & & y_1^1 & \\ p'_{13} & x_2^3 & & . \end{array}$$

The second derivatives corresponding to new tangent vectors come from:

$$\begin{array}{cccc}
p_1'' & & & z_2^2 \\
p_2'' & & & \\
p_3'' & x_1^1 & & -z_2^2 \\
p_4'' & x_1^3 & & \\
p_5'' & & & -z_2^2 + z_4^2 \\
p_6'' & x_1^1 & -y_1^2 & \\
p_7'' & & & \\
p_8'' & & & z_2^2 \\
p_9'' & & & \\
p_{10}'' & x_1^3 & & \\
p_{11}'' & & & \\
p_{12}'' & x_1^3 & -y_1^2 & z_4^2 \\
p_{13}'' & & & .
\end{array}$$

Note that p_3, p_4 lie on $L_{(\gamma, \mu)}$ and p_2, p_5 lie on $L_{(\alpha, \mu^*)}$ and p_1 is chosen exactly so it cancels $p_2 + \dots + p_5$, just as with T_{BCLR} . Note further that $p_6, p_7 \in S_{(\gamma, \omega^*)}$, $p_8, p_9 \in S_{(\alpha, \nu)}$, $p_{10}, p_{11} \in T_{(\gamma, \omega^*)}$, and $p_{12}, p_{13} \in T_{(\alpha, \nu)}$.

Remark 9.1. The permutation $U \rightarrow U$ (and its induced action $U^* \rightarrow U^*$) exchanging $u_1 \leftrightarrow u_3$ and $u_2 \leftrightarrow u_4$ preserves $M_{(4,2,2)}$. In the algorithm it switches the role of the S 's and T 's and fixes the L 's.

The first order derivatives are as follows:

p_1, \dots, p_5 from the $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ contribute the following terms:

$$\begin{array}{ll}
p_1 & -x_2^1 \otimes (-y_1^1 + y_2^2) \otimes (-z_1^1 + z_3^1) - (x_2^2 + x_4^2) \otimes (-y_1^1 + y_2^2) \otimes z_4^1 \\
p_2 & x_2^1 \otimes y_2^2 \otimes (-z_1^1 + z_3^1) + x_2^2 \otimes y_1^1 \otimes (-z_1^1 + z_3^1) - x_2^2 \otimes y_2^2 \otimes (-z_4^1 + z_1^2) \\
p_3 & (x_2^1 - x_1^2) \otimes y_1^1 \otimes z_1^1 \\
p_4 & -(x_1^4 + x_2^1) \otimes y_1^1 \otimes z_3^1 - (x_2^2 + x_4^2) \otimes y_1^1 \otimes z_3^1 - (x_2^2 + x_4^2) \otimes y_1^1 \otimes z_4^1 \\
p_5 & x_2^4 \otimes y_2^2 \otimes (z_4^1 - z_3^2)
\end{array}$$

These sum to $-x_2^2 \otimes y_1^1 \otimes z_1^1 - x_2^2 \otimes y_2^2 \otimes z_1^2 - x_1^2 \otimes y_1^1 \otimes z_1^1 - x_1^4 \otimes y_1^1 \otimes z_3^1 - x_2^4 \otimes y_1^1 \otimes z_3^1 - x_2^4 \otimes y_2^2 \otimes z_3^2$. The points from $S_{(\gamma, \omega^*)}$ and $T_{(\gamma, \omega^*)}$ contribute:

$$\begin{array}{ll}
p_6 & x_2^2 \otimes y_2^2 \otimes z_1^2 \\
p_7 & -x_2^2 \otimes y_2^1 \otimes z_2^1 \\
p_8 & -x_1^2 \otimes (-y_1^1 + y_2^2) \otimes z_1^1 - x_1^2 \otimes y_2^1 \otimes z_2^1 \\
p_9 & (x_1^2 + x_2^2) \otimes y_1^1 \otimes z_1^1 + (x_1^2 + x_2^2) \otimes y_2^1 \otimes z_2^1.
\end{array}$$

These sum to $x_2^2 \otimes -y_1^2 \otimes -z_1^1 + x_2^2 \otimes y_2^2 \otimes z_1^2 + x_1^2 \otimes y_1^1 \otimes z_1^1$.

$T_{(\gamma, \omega^*)}$ and $T_{(\alpha, \nu)}$ contribute:

$$\begin{array}{ll}
p_{10} & x_2^3 \otimes y_2^1 \otimes (-z_3^1 + z_3^2) - x_2^4 \otimes y_1^1 \otimes (-z_3^1 + z_3^2) \\
p_{11} & -x_2^3 \otimes y_2^1 \otimes z_3^2 + x_2^4 \otimes (y_1^1 + y_2^2) \otimes z_3^2 \\
p_{12} & x_1^4 \otimes y_1^1 \otimes z_3^1 \\
p_{13} & x_2^3 \otimes y_2^1 \otimes z_3^1.
\end{array}$$

These sum to $x_1^4 \otimes y_1^1 \otimes z_3^1 + x_2^4 \otimes y_1^1 \otimes z_3^1 + x_2^4 \otimes y_2^2 \otimes z_3^2$.

The second order terms are as follows:

Terms from L_1, L_2 in the second fundamental form are:

$$\begin{array}{ll}
 p_1 & x_2^1 \otimes (-y_1^1 + y_2^2) \otimes z_4^1 \\
 p_2 & -x_2^1 \otimes y_1^2 \otimes (-z_1^1 + z_3^1) + x_2^1 \otimes y_2^2 \otimes (-z_4^1 + z_1^2) + x_2^2 \otimes y_1^2 \otimes (-z_4^1 + z_1^2) \\
 p_3 & \\
 p_4 & (x_1^4 + x_2^1) \otimes y_1^2 \otimes z_3^1 + (x_1^4 + x_2^1) \otimes y_1^1 \otimes z_4^1 + (x_2^2 + x_2^4) \otimes y_1^2 \otimes z_4^1 \\
 p_5 & .
 \end{array}$$

These sum to $x_2^1 \otimes y_2^2 \otimes z_1^2 + x_2^1 \otimes y_1^2 \otimes z_1^1 + x_2^2 \otimes y_1^2 \otimes z_1^2 + x_1^4 \otimes y_1^2 \otimes z_3^1 + x_1^4 \otimes y_1^1 \otimes z_4^1 + x_2^4 \otimes y_1^2 \otimes z_4^1$.
The t^2 terms from tangent vectors to L_1 and L_2 are:

$$\begin{array}{ll}
 p_1 & (x_2^2 + x_2^4) \otimes (-y_1^1 + y_2^2) \otimes z_2^2 \\
 p_2 & \\
 p_3 & x_1^1 \otimes y_1^1 \otimes z_1^1 + (x_2^2 + x_2^4) \otimes y_1^1 \otimes z_2^2 \\
 p_4 & x_1^3 \otimes y_1^1 \otimes z_3^1 \\
 p_5 & x_2^4 \otimes y_2^2 \otimes (-z_2^2 + z_4^2)
 \end{array}$$

These sum to $x_1^1 \otimes y_1^1 \otimes z_1^1 + x_2^2 \otimes y_2^2 \otimes z_2^2 + x_2^4 \otimes y_2^2 \otimes z_4^2 + x_1^3 \otimes y_1^1 \otimes z_3^1$.
Terms from $S_{(\gamma, \omega^*)}, S_{(\alpha, \nu)}$ in the second fundamental form are:

$$\begin{array}{ll}
 p_6 & \\
 p_7 & \\
 p_8 & -x_1^2 \otimes (-y_1^1 + y_1^2) \otimes z_2^1 \\
 p_9 & (x_1^2 + x_2^2) \otimes y_1^2 \otimes z_2^1
 \end{array}$$

These sum to $x_1^2 \otimes y_1^1 \otimes z_2^1 + x_2^2 \otimes y_1^2 \otimes z_2^1$.

The $S_{(\gamma, \omega^*)}, S_{(\alpha, \nu)}$ t^2 terms from tangent spaces are:

$$\begin{array}{ll}
 p_6 & x_1^1 \otimes y_2^1 \otimes z_1^2 - x_2^2 \otimes y_1^2 \otimes z_1^2 \\
 p_7 & \\
 p_8 & x_1^2 \otimes y_2^1 \otimes z_2^2 \\
 p_9 &
 \end{array}$$

These sum to $x_1^2 \otimes y_2^1 \otimes z_2^2 - x_2^2 \otimes y_1^2 \otimes z_1^2 + x_1^1 \otimes y_2^1 \otimes z_1^2$.

Terms from $T_{(\gamma, \omega^*)}$ and $T_{(\alpha, \nu)}$ in the second fundamental form are:

$$\begin{array}{ll}
 p_{10} & -x_2^3 \otimes y_1^2 \otimes (-z_3^1 + z_3^2) \\
 p_{11} & x_2^3 \otimes (y_1^2 + y_2^2) \otimes z_3^2 \\
 p_{12} & \\
 p_{13} &
 \end{array}$$

These sum to $x_2^3 \otimes y_1^2 \otimes z_3^1 + x_2^3 \otimes y_2^2 \otimes z_3^2$.

The $T_{(\gamma, \omega^*)}$ and $T_{(\alpha, \nu)}$ t^2 terms from tangent spaces are:

$$\begin{array}{ll}
 p_{10} & x_1^3 \otimes y_2^1 \otimes (-z_3^1 + z_3^2) \\
 p_{11} & \\
 p_{12} & x_1^3 \otimes y_2^1 \otimes z_3^1 - x_1^4 \otimes y_1^2 \otimes z_3^1 + x_1^4 \otimes y_2^1 \otimes z_4^2 \\
 p_{13} &
 \end{array}$$

These sum to $x_1^3 \otimes y_2^1 \otimes z_3^2 - x_1^4 \otimes y_1^2 \otimes z_3^1 + x_1^4 \otimes y_2^1 \otimes z_4^2$.

10. BRIEF REMARKS ON SMIRNOV'S BORDER RANK 20 ALGORITHM FOR $M_{(3,3,3)}$

The [10, Table 6] border rank algorithm for $M_{(3,3,3)}$ expressed as a tensor is:

$$\begin{aligned}
p_1(t) &= t^{-6}(t^3 x_1^1 - t^6 x_3^1 + t x_1^3 + x_3^3) \otimes (t^2 y_1^1 + t^3 y_2^1 + t y_1^2 + y_1^3 + 2t^5 y_3^3) \otimes (z_1^2 - t^4 z_2^1 + t^6 z_3^1) \\
p_2(t) &= t^{-5}(t^5 x_3^1 + x_1^2 - t x_2^2 - t^3 x_3^3) \otimes (t^4 y_3^2 + y_3^3) \otimes (-t^3 z_1^1 + z_1^3 - t^3 z_2^1 + t^2 z_2^2 + t^3 z_2^3 - t^2 z_3^3) \\
p_3(t) &= t^{-5}(-t x_1^2 + t^2 x_2^2 - x_3^3) \otimes (y_1^2 + t y_2^2 - t^3 y_3^3) \otimes (t^3 z_1^1 - z_1^3 + (t^2 - t^6) z_2^1 + t^2 z_3^3) \\
p_4(t) &= t^{-5}(-t^2 x_1^1 + t^3 x_2^1 - x_1^3 + t x_2^3) \otimes (t^2 y_1^1 + t y_1^2 + y_1^3) \otimes (-t z_1^1 + z_1^2 + t^6 z_3^1) \\
p_5(t) &= t^{-6}(-t^5 x_3^1 + x_1^2 + x_3^3) \otimes (t^2 y_1^1 + t^3 y_2^1 + t y_1^2 + y_1^3 + t y_2^3 + t^5 y_3^3) \otimes (-z_1^2 + t^4 z_2^1) \\
p_6(t) &= t^{-6}(x_1^3 - t x_2^3) \otimes (t^4 y_2^1 + y_1^2) \otimes (t^2 z_1^1 - t^2 z_1^2 - z_1^3 - (t^4 + t^5) z_3^1 + t^4 z_3^2 + t^2 z_3^3) \\
p_7(t) &= t^{-4}(t^3 x_1^1 - 2t^4 x_2^1 + x_1^3 - t x_2^3) \otimes (t^2 y_1^1 + t^4 y_3^1 + y_1^2) \otimes (-z_1^1) \\
p_8(t) &= t^{-6}(t^3 x_1^1 - t^5 x_3^1 - x_1^2 + t^2 x_3^2 + t x_1^3) \otimes (t^3 y_3^1 + y_1^3) \otimes (z_1^3 + (t^4 - t^3) z_2^1) \\
p_9(t) &= t^{-5}(-x_1^2 + t x_2^2) \otimes (t^2 y_2^2 + y_3^3) \otimes (-t^3 z_1^1 + z_1^3 - t^3 z_2^1 + (t^2 - t^5) z_2^2 - t^2 z_3^3) \\
p_{10}(t) &= t^{-5}(x_2^3) \otimes (y_1^2 + t^4 y_3^1 + t y_2^2) \otimes (t^3 z_1^1 - t^2 z_1^2 - z_1^3 + t^3 z_2^1 + t^4 z_3^2 + t^2 z_3^3) \\
p_{11}(t) &= t^{-4}(x_1^2 - t^2 x_3^2) \otimes (t y_3^1 + y_2^2 + y_3^3) \otimes (-t^3 z_2^1 + z_1^3 + t^3 z_3^3) \\
p_{12}(t) &= t^{-6}(-x_1^2 + t^4 x_2^1 + t^2 x_3^2 + t^2 x_3^3) \otimes (-t^2 y_3^2 + y_1^3) \otimes (t^2 z_2^1 - z_1^3) \\
p_{13}(t) &= t^{-6}(-t^2 x_1^2 + t^3 x_2^2 + x_1^3 - t x_2^3) \otimes y_1^2 \otimes (-t^3 z_1^1 + z_1^3 + t^6 z_2^1 - t^2 z_3^3) \\
p_{14}(t) &= t^{-5}(-t^2 x_1^1 + t^3 x_2^1 + t^4 x_3^1 - x_1^3 + t x_2^3) \otimes (y_1^3) \otimes (t z_1^1 + z_1^3 - t^3 z_2^1) \\
p_{15}(t) &= t^{-6}(x_3^3) \otimes (y_2^3 - t^4 y_3^3) \otimes (t z_1^2 - z_1^3 - t^5 z_2^1 - t^2 z_2^2 - t^3 z_2^3 + t^6 z_3^2) \\
p_{16}(t) &= t^{-6}(-t^4 x_2^1 + x_1^2 - t^2 x_3^2) \otimes (t^2 y_1^1 + t y_1^2 - t^2 y_2^2 + y_2^3) \otimes (z_1^2) \\
p_{17}(t) &= t^{-4}(x_1^3) \otimes (-t^2 y_2^1 + t^4 y_3^1 + y_1^2 - 2t^4 y_3^3) \otimes (z_1^2 + t^3 z_3^1 - t^2 z_3^2) \\
p_{18}(t) &= t^{-2}(-x_1^2 + t x_2^2 + t^2 x_3^2) \otimes (t y_3^2 + y_3^3) \otimes (t^2 z_2^2 + z_2^3) \\
p_{19}(t) &= t^{-6}(-t^2 x_1^2 + t^4 x_3^2 + x_3^3) \otimes (y_2^3) \otimes (z_1^3 + t^2 z_2^2 + t^3 z_3^3) \\
p_{20}(t) &= t^{-5}(x_1^2) \otimes (t^2 y_2^1 + t y_2^2 + y_2^3 + t^4 y_3^3) \otimes (z_1^2 + t^3 z_2^2).
\end{aligned}$$

The limit points are

p_1	x_3^3	y_1^3	z_1^2
$p_2 = p_9$	x_1^2	y_3^3	z_1^3
$p_3 = p_{10}$	$-x_2^3$	y_1^2	$-z_1^3$
p_4	$-x_1^3$	y_1^3	z_1^2
p_5	$x_1^2 + x_3^3$	y_1^3	$-z_1^2$
p_6	x_1^3	y_1^2	$-z_1^3$
p_7	x_1^3	y_1^2	$-z_1^1$
p_8	$-x_1^2$	y_1^3	z_1^3
p_{11}	x_1^2	$y_3^2 + y_2^3$	z_1^3
p_{12}	$-x_1^2$	y_2^3	$-z_1^3$
p_{13}	x_1^3	y_1^2	z_1^3
p_{14}	$-x_1^3$	y_1^3	z_1^3
$p_{15} = p_{19}$	x_3^3	y_2^3	$-z_1^3$
$p_{16} = p_{20}$	x_1^2	y_2^3	z_1^2
p_{17}	x_1^3	y_1^2	z_1^2
p_{18}	$-x_1^2$	y_3^3	z_2^3

In addition to the duplication of points, the limit points are in a very degenerate configuration. For example all the z points lie on a \mathbb{P}^2 of type ω^* , all the x points lie on a \mathbb{P}^3 and all the y points on a \mathbb{P}^4 .

11. REMARKS ON THE UNIQUENESS OF THE BCLR BORDER RANK ALGORITHMS

Let $T = T_1 + \cdots + T_r$ be a rank r expression for a tensor $T \in A \otimes B \otimes C$. The tensor T is said to be *identifiable* if the $[T_j]$ are unique, i.e., T is not in the span of any other collection of r rank one tensors (up to scale). For border rank algorithms it will be more useful to define a weaker notion of identifiability: we will say T is *Grassmann identifiable* if $\langle T_1, \dots, T_r \rangle$ is the unique r -plane spanned by rank one tensors that contains T , and *Grassmann border identifiable* if there exists a unique $E \in G(r, A \otimes B \otimes C)$ that is the limit of some $\langle T_1(t), \dots, T_r(t) \rangle$ with $T_j(t)$ of rank one.

Tensors with symmetry are rarely Grassmann border identifiable because their symmetry group acts on the Grassmannian and will move the algorithm to other algorithms. In what follows we discuss the action of the symmetry group on the limiting 5-plane E^{BCLR} for the BCLR tensor, and the limiting 10-plane for the sum of two BCLR-type tensors glued together to form a border rank algorithm for $M_{(3,2,2)}$, which we denote by \tilde{E}^{BCLR} .

We expect the following information to be useful in constructing new algorithms for $M_{(n,2,2)}$.

The role x_1^1 in a BCLR-type algorithm can be played by any element of $\text{Seg}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$, so the glued together algorithms come in families parametrized by this Segre variety.

The choice of an element to blank out determines a split $U = \mathbb{C}^1 \oplus \mathbb{C}^{n-1}$ and $V = \mathbb{C}^1 \otimes \mathbb{C}^1$. The subgroup preserving such a splitting is $G_U \times T_V \times SL(W)$, where $T_V \subset SL(V)$ denotes the diagonal matrices and

$$G_U = \begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix} \cap SL(U)$$

where the blocking is $(1, n-1) \times (1, n-1)$.

A border rank algorithm for $M_{(n,2,2)}$ obtained from two BCLRS-type algorithms has a further splitting $\mathbb{C}^{n-1} = \mathbb{C}^{m-1} \oplus \mathbb{C}^{n-m}$.

The following was shown via a computer calculation of the Lie algebra of the stabilizers by F. Gesmundo:

Proposition 11.1. $\tilde{E}^{BCLR} \in G(10, A \otimes B \otimes C)$ has a 10 dimensional orbit under $SL(U) \times SL(V) \times SL(W)$. The connected component of the identity of its stabilizer is $T_U \times T_V \times T_W$.

$E^{BCLR} \in G(5, A \otimes B \otimes C)$ has a 2-dimensional orbit under $G_U \times T_V \times SL(W)$. The connected component of the identity of its stabilizer is $T_U \times T_V \times T_W$.

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